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COMPACTNESS OF MARKOV AND SCHRÖDINGER SEMI-GROUPS: A PROBABILISTIC APPROACH

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Abstract
It is proved if an irreducible, strong Feller symmetric Markov process possesses a tightness property, then its semi-group is an L^2-compact operator. In this paper, applying this fact, we prove probabilistically the compactness of Dirichlet-Laplacians and Schrödinger operators.

1. Introduction

Let E be a locally compact separable metric space and m a positive Radon measure on E with full support. Let X be an m-symmetric Markov process on E. We assume that X is irreducible and has strong (resolvent) Feller property. Moreover, we assume that X possesses the *tightness property*, i.e., for any $\epsilon > 0$ there exists a compact set K such that $\sup_{x\in E} R_1 1_{K^c}(x) \le \epsilon$. Here R_1 is the 1-resolvent of X and 1_{K^c} is the indicator function of the complement of K. When X has these properties, we say in this paper that X belongs to Class (T). One of the authors proved in [14] that if X belongs to Class (T), its semi-group turns out to be a compact operator on $L^2(E;m)$ (Theorem 2.1). In this paper, we apply this criterion to Dirichlet Laplacians Δ_D and Schrödinger operators $\Delta - V$ with positive potential and show probabilistically the compactness of these operators.

Let $X = (\mathbb{P}_x, B_t)$ be the Brownian motion on $\mathbb{R}^d$ and X^D the absorbing Brownian motion on a domain D. We then prove that if $D \subset \mathbb{R}^d$ satisfies $\lim_{x\in D,|x|\to\infty} m(D \cap B(x,1)) = 0$, then X^D is in Class (T) and consequently its semi-group is compact. Here m denotes the Lebesgue measure and $B(x,R)$ the ball centered at x with radius R. If x is the origin 0, we write $B(R)$ for $B(0,R)$.

We denote by $\mathcal{B}_0$ the set of Borel sets B such that $\lim_{|x|\to\infty} m(B \cap B(x,1)) = 0$. In [8], a Borel set in $\mathcal{B}_0$ is said to be *thin at infinity*. Let V be a positive Borel function on $\mathbb{R}^d$ locally in the Kato class. Let $X^V = (\mathbb{P}^V_x, B_t)$ be the subprocess defined by $\mathbb{P}^V_x(d\omega) = \exp\left(-\int_0^t V(B_s(\omega))ds\right)\mathbb{P}_x(d\omega)$. We show that if the set $D_M := \{x \in \mathbb{R}^d \mid V(x) \le M\}$ belongs to $\mathcal{B}_0$ for any $M > 0$, then X^V is in Class (T) and its semi-group, Schrödinger semi-group of $-\Delta + V$, is compact. This fact is proved in [11], [8] analytically, while it is done in this paper probabilistically; the key to the proof of this fact is to show that the condition on V implies the tightness property of X^V.

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We apply Theorem 2.1 to time changed processes. Let X be an irreducible symmetric Markov process with strong Feller property. We assume, in addition, that X is transient. We then see that for a Green-tight measure μ with full fine support, the time-changed process $\check{X}$ by A_t^μ, the positive continuous additive functional in the Revez correspondence to μ, belongs to Class (T). As a results, the space $(\check{\mathcal{F}}, \check{\mathcal{E}}_1 = (\check{\mathcal{E}} + (\ ,\ )_\mu))$ is compactly embedded in $L^2(E;\mu)$, where $(\check{\mathcal{E}}, \check{\mathcal{F}})$ is the Dirichlet form generated by $\check{X}$. Moreover, let $(\mathcal{F}_e, \mathcal{E})$ be the extended Dirichlet space $(\mathcal{F}_e, \mathcal{E})$ associated with X, which turns out to be a Hilbert space under the condition for X being transient. We then see that $\mathcal{F}_e$ is identical with $\check{\mathcal{F}}$, $\mathcal{E}$ is equivalent to $\check{\mathcal{E}}_1$ and thus $(\mathcal{F}_e, \mathcal{E})$ is compactly embedded in $L^2(E;\mu)$. Therefore, we can conclude that for any Green-tight measure μ, the extended Dirichlet space $(\mathcal{F}_e, \mathcal{E})$ is compactly embedded in $L^2(E;\mu)$. This fact says that if μ is Green-tight with respect to 1-resolvent, then $(\mathcal{F}, \mathcal{E}_1 = (\mathcal{E} + (\ ,\ )_m))$ is compactly embedded in $L^2(E;\mu)$.

Applying this result to the Brownian motion, we see that if $B \in \mathcal{B}(\mathbb{R}^d)$ satisfies that the measure $1_B dx$ is Green-tight, then 1-order Sobolev space $H^1(\mathbb{R}^d)$ is compactly embedded in $L^2(\mathbb{R}^d; 1_B dx)$. We see from [4, Lemma 6.11] that for a domain B, this is also necessary.

2. Preliminaries

Let E be a locally compact separable metric space, $E_\Delta = E \cup \{\Delta\}$ the one point compactification of E, and m a positive Radon measure on E with full support. Let $X = (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, X_t, \mathbb{P}_x, \zeta)$ be an m-symmetric Borel right process having left limits on $(0, \zeta)$. Here ζ is the lifetime $\zeta(\omega) = \inf\{s \geq 0 \mid X_s(\omega) = \Delta\}$ and Ω is specifically taken to be the space of all right continuous functions from $[0, \infty]$ into E_Δ with $\omega(t) = \Delta$ for any $t \geq \zeta(\omega) = \inf\{s \geq 0 \mid w(s) = \Delta\}$ and $\omega(\infty) = \Delta$. The random variable ζ is called the lifetime which can be finite and X_t is defined by $X_t(\omega) = \omega(t)$ for $\omega \in \Omega,\ t \geq 0$. $\{\mathcal{F}_t\}_{t\geq 0}$ is the minimal (augmented) admissible filtration.

Let $\{p_t\}_{t\geq 0}$ be the semi-group of X, $p_t f(x) = \mathbb{E}_x(f(X_t))$. By Lemma 1.4.3 in [5], $\{p_t\}_{t\geq 0}$ uniquely determines a strongly continuous Markovian semi-group $\{T_t\}_{t\geq 0}$ on $L^2(E;m)$. We define the Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ on $L^2(E;m)$ generated by X:

$$
\begin{cases} \mathcal{F} = \left\{ u \in L^2(E;m) \mid \lim\limits_{t\to 0} \dfrac{1}{t}(u - T_t u, u)_m < \infty \right\} \\ \mathcal{E}(u,v) = \lim\limits_{t\to 0} \dfrac{1}{t}(u - T_t u, v)_m \ \text{ for } u, v \in \mathcal{F}. \end{cases} \tag{2.1}
$$

We denote by $\mathcal{F}_e$ the family of m-measurable functions u on X such that $|u| < \infty$ m-a.e. and there exists an $\mathcal{E}$-Cauchy sequence $\{u_n\}$ of functions in $\mathcal{F}$ such that $\lim_{n\to\infty} u_n = u$ m-a.e. We call $\mathcal{F}_e$ the *extended Dirichlet space* of $(\mathcal{E}, \mathcal{F})$. Every $u \in \mathcal{F}_e$ has a quasi-continuous version $\widetilde{u}$ ([5, Theorem 2.1.3]). In the sequel, we always assume that every function $u \in \mathcal{F}_e$ is represented by its quasi-continuous version.

Let us denote by $\{R_\alpha\}_{\alpha>0}$ the resolvent of X,

$$
R_\alpha f(x) = \mathbb{E}_x\left(\int_0^\infty e^{-\alpha t} f(X_t) dt \right), \quad f \in \mathcal{B}_b(E),
$$

where $\mathcal{B}_b(E)$ is the space of bounded Borel functions on E. We now make three assumptions on X:

I. (**Irreducibility**) If a Borel set A is p_t-invariant, i.e., $\int_A p_t 1_{A^c} dm = 0$ for any $t > 0$, then A satisfies either $m(A) = 0$ or $m(A^c) = 0$. Here 1_{A^c} is the indicator function of the complement of A.

II. (**Resolvent Strong Feller Property**) $R_\alpha(\mathcal{B}_b(E)) \subset C_b(E)$, $\alpha > 0$, where $C_b(E)$ is the space of bounded continuous functions.

III. (**Tightness Property**) For any $\epsilon > 0$, there exists a compact set K such that $\sup_{x\in E} R_1 1_{K^c}(x) \le \epsilon$. Here 1_{K^c} is the indicator function of the complement of K.

We here say that a Markov process belongs to **Class (T)** if it possess the properties I, II, III.

REMARK 2.1. (i) If $R_1 1 \in C_\infty(E)$, then X is explosive and satisfies the assumption III. In fact, it follows from the maximum property that

$$\sup_{x\in E} R_1 1_{K^c}(x) = \sup_{x\in K^c} R_1 1_{K^c}(x) \le \sup_{x\in K^c} R_1 1(x).$$

Here $C_\infty(E)$ is the set of continuous functions vanishing at infinity.

(ii) If $C_\infty(E)$ is invariant under R_1, $R_1(C_\infty(E)) \subset C_\infty(E)$, then $R_1 1 \in C_\infty(E)$ is equivalent to III. In fact, assume III. For a compact set K, take a positive function $g \in C_\infty(E)$ such that $1_K \le g$. We then see from the invariance of $C_\infty(E)$ that $0 \le \lim_{x\to\infty} R_1 1_K(x) \le \lim_{x\to\infty} R_1 g(x) = 0$. Hence for any $\epsilon > 0$ there exists a compact set K such that

$$\limsup_{x\to\infty} R_1 1(x) \le \limsup_{x\to\infty} R_1 1_K(x) + \limsup_{x\to\infty} R_1 1_{K^c}(x) \le \sup_{x\in E} R_1 1_{K^c}(x) \le \epsilon,$$

which implies $R_1 1 \in C_\infty(E)$. Hence, if $C_\infty(E)$ is invariant under R_1 and X is conservative, $R_1 1 = 1$, then X does not have the tightness property, in particular, the Ornstein-Uhlenbeck process does not.

(iii) Assume that m is finite, $m(E) < \infty$ and that $\{p_t\}_{t\ge 0}$ is ultra-contractive, $\|p_t\|_{1,\infty} = c_t < \infty$ for any $t > 0$. Here $\|\cdot\|_{1,\infty}$ is the operator norm from $L^1(E;m)$ to $L^\infty(E;m)$. Note that c_t is non-increasing because $\|p_t\|_{1,\infty} \le \|p_s\|_{1,\infty} \cdot \|p_{t-s}\|_{\infty,\infty} \le \|p_s\|_{1,\infty}$ for $0 < s < t$. We then see that X has the tightness property III. Indeed, for any $\delta > 0$ and a compact set $K \subset \mathbb{R}^d$

$$R_1 1_{K^c}(x) \le \int_0^\delta e^{-t} p_t 1_{K^c}(x)dt + \int_\delta^\infty e^{-t} p_t 1_{K^c}(x)dt \le (1 - e^{-\delta}) + c_\delta \cdot m(K^c).$$

Hence for any $\varepsilon > 0$ $\|R_1 1_{K^c}\|_\infty < \varepsilon$, if $\delta > 0$ and a compact set K satisfy $1-\exp(-\delta) < \varepsilon/2$ and $c_\delta \cdot m(K^c) < \varepsilon/2$.

It follows from the assumption II that the resolvent kernel is absolutely continuous with respect to m,

$$R_\beta(x, dy) = R_\beta(x, y)m(dy), \quad \text{for each } \alpha > 0,\ x \in E.$$

As a result, the transition probability $p_t(x, dy)$ is also absolutely continuous with respect to m,

$$p_t(x, dy) = p_t(x, y)m(dy) \ \text{ for each } t > 0,\ x \in E$$

([5, Theorem 4.2.4]). By [5, Lemma 4.2.4] the density $R_\beta(x, y)$ is assumed to be a non-negative Borel function such that $R_\beta(x, y)$ is symmetric and β-excessive in x and in y. Under the absolute continuity condition, "quasi-everywhere" statements are strengthened to "everywhere" ones.

One of the authors proved the next theorem ([14, Theorem 4.1]).

Theorem 2.1 ([14])**.** *If a Markov process X is in Class (T), then its semi-group p_t is compact on $L^2(E;m)$.*

We denote by S_{00} the set of positive Borel measures μ such that $\mu(E) < \infty$ and $R_1\mu(x)(= \int_E R_1(x,y)\mu(dy))$ is uniformly bounded in $x \in E$. A positive Borel measure μ on E is said to be *smooth* if there exists a sequence $\{E_n\}_{n=1}^{\infty}$ of Borel sets increasing to E such that $1_{E_n} \cdot \mu \in S_{00}$ for each n and

$$\mathbb{P}_x\left(\lim_{n\to\infty} \sigma_{E\setminus E_n} \geq \zeta\right) = 1, \quad \forall x \in E,$$

where $\sigma_{E\setminus E_n}$ is the first hitting time of $E \setminus E_n$. The totality of smooth measures is denoted by S_1.

If an additive functional $\{A_t\}_{t\geq 0}$ is positive and continuous with respect to t for each $\omega \in \Omega$, it is said to be a *positive continuous additive functional* (PCAF in abbreviation). By [5, Theorem 5.1.7][1], there exists a one-to-one correspondence between positive smooth measures and PCAF's (**Revuz correspondence**): for each smooth measure μ, there exists a unique PCAF $\{A_t\}_{t\geq 0}$ such that for any positive Borel function f on E and γ-excessive function h $(\gamma \geq 0)$, that is, $e^{-\gamma t} p_t h \leq h$,

$$\lim_{t\to 0} \frac{1}{t}\mathbb{E}_{h\cdot m}\left(\int_0^t f(X_s)dA_s\right) = \int_E f(x)h(x)\mu(dx). \tag{2.2}$$

Here $\mathbb{E}_{h\cdot m}(\,\cdot\,) = \int_E \mathbb{E}_x(\,\cdot\,)h(x)m(dx)$. We denote by A_t^{μ} the PCAF corresponding to the smooth measure μ.

We now introduce two classes of positive smooth measures which play a crucial role.

DEFINITION 2.1. (i) A positive measure $\mu \in S_1$ is said to be in the *Kato class* (in notation, $\mu \in \mathcal{K}$), if

$$\lim_{\beta\to\infty} \sup_{x\in E} \int_E R_\beta(x,y)d\mu(y) = 0.$$

A positive measure $\mu \in S_1$

(ii) Suppose X is transient. A measure $\mu \in \mathcal{K}$ is said to be *Green-tight* (in notation, $\mu \in \mathcal{K}_\infty(R)$), if for any $\epsilon > 0$ there exists a compact set K such that

$$\sup_{x\in E} \int_{K^c} R(x,y)d\mu(y) \leq \epsilon.$$

If the measure $\mu(dx) = V(x)m(dx)$ is in $\mathcal{K}$ (resp. $\mathcal{K}_\infty$), we also denote $V \in \mathcal{K}$ (resp. $\mathcal{K}_\infty$).

Note that if X is transient, then $(\mathcal{F}_e, \mathcal{E})$ is a Hilbert space. The next theorem is proved by Stollmann-Voigt [13].

[1] In [5], the measure μ (resp. PCAF A_t) is said to be a *smooth measure in the strict sense* (resp. a *PCAF in the strict sense*). We treat only smooth measures in the strict sense and PCAF's in the strict sense, and omit the term "in the strict sense".

Theorem 2.2. *For $\mu \in \mathcal{K}_\infty(R)$*

$$\int_E u^2(x)\mu(dx) \leq \|R\mu\|_\infty \cdot \mathcal{E}(u,u), \quad u \in \mathcal{F}_e. \tag{2.3}$$

Here, $R\mu(x) = \int_E R(x,y)d\mu(y)$.

Note that $\|R\mu\|_\infty$ is finite by [2, Proposition 2.2]. Let $\check{X} = (\check{\mathbb{P}}, \check{X}_t)$ be the time-changed process, that is, $\check{\mathbb{P}}_x = \mathbb{P}_x$, $\check{X}_t = X_{\tau_t}$, $\tau_t = \inf\{s > 0 : A_s^\mu > t\}$. Define

$$F = \{x \in X : \mathbb{P}_x(\tau_0 = 0) = 1\}.$$

We call F the *fine support* of μ. Note that the 0-resolvent $\check{R}$ of $\check{X}$ is written as

$$\check{R}f(x) = \int_F R(x,y)f(x)d\mu(x), \quad f \in L^2(F;\mu),$$

We then see from (2.3) that for $\mu \in \mathcal{K}_\infty(R)$, $(\mathcal{F}_e, \mathcal{E})$ is continuously embedded in $L^2(E;\mu)$ and so $\check{R}$ is a bounded operator on $L^2(F;\mu)$.

Theorem 2.3 ([14]). *Assume that a Markov process X satisfies I and II. If X is transient, then for $\mu \in \mathcal{K}_\infty(R)$, $(\mathcal{F}_e, \mathcal{E})$ is compactly embedded in $L^2(E;\mu)$.*

Theorem 2.3 is an extension of Theorem 2.1. Indeed, $(\mathcal{F}, \mathcal{E}_1)$ is a transient regular Dirichlet space and its extended Dirichlet space equals $(\mathcal{F}, \mathcal{E}_1)$. Notice that the 1-resolvent R_1 is identical with the 0-resolvent of $(\mathcal{F}, \mathcal{E}_1)$. We then see from Theorem 2.3 that if μ is Green-tight with respect to the 1-resolvent R_1 (in notation, $\mu \in \mathcal{K}_\infty(R_1)$), then $(\mathcal{F}, \mathcal{E}_1)$ is compactly embedded in $L^2(E;\mu)$. It is known in [2, Theorem 4.2] that if X is in Class (T), then m belongs to $\mathcal{K}_\infty(R_1)$. We then obtain Theorem 2.1 because the semi-group p_t is compact if and only if $(\mathcal{F}, \mathcal{E}_1)$ is compactly embedded in $L^2(E;m)$.

Corollary 2.1. *Assume that a Markov process X satisfies I and II. If μ is a smooth measure in $\mathcal{K}_\infty(R_1)$, then $(\mathcal{F}, \mathcal{E}_1)$ is compactly embedded in $L^2(E;\mu)$. In particular, if X is in Class (T), $(\mathcal{F}, \mathcal{E}_1)$ is compactly embedded in $L^2(E;m)$.*

Theorem 2.3 and Corollary 2.1 tell us that the 0-resolvent and 1-resolvent of $\check{X}$ define compact operators on $L^2(F;\mu)$ respectively.

3. Dirichlet Laplacian

In this section, we deal with the Brownian motion $X = (\mathbb{P}_x, B_t)$ on $\mathbb{R}^d$ and give, as an application of Theorem 2.1, a sufficient condition for the compactness of semi-groups of Dirichlet-Laplacians.

Lemma 3.1. *Let p_t be the semi-group of the Brownian motion. Then*

$$\|p_t\|_{p,\infty} \leq \frac{C}{t^{d/(2p)}}, \quad p \geq 1,$$

where $\|\cdot\|_{p,\infty}$ is the operator norm from $L^p(\mathbb{R}^d)$ to $L^\infty(\mathbb{R}^d)$.

Proof. Note that for $f \in L^p(\mathbb{R}^d)$,

$$|p_t f(x)| \leq (p_t|f|^p(x))^{1/p}.$$

Hence we have, on account of $\|p_t\|_{1,\infty} \le C/t^{d/2}$

$$\|p_t f\|_\infty \le \|p_t|f|^p\|_\infty^{1/p} \le \|p_t\|_{1,\infty}^{1/p} \cdot \||f|^p\|_1^{1/p} = \|p_t\|_{1,\infty}^{1/p} \cdot \|f\|_p \le \frac{C}{t^{d/(2p)}} \cdot \|f\|_p.$$

□

Let $\mathcal{D}$ the set of domains in $\mathbb{R}^d$. We set

$$\mathcal{D}_0 = \left\{ D \in \mathcal{D} \mid \lim_{x \in D, |x| \to \infty} m(D \cap B(x,1)) = 0 \right\},$$

where m denotes the Lebesgue measure on $\mathbb{R}^d$.

Denote by τ_B be the first exit time from a Borel set B, $\tau_B = \inf\{t > 0 \mid B_t \notin B\}$.

Lemma 3.2. *If a domain D belongs to $\mathcal{D}_0$, then $\lim_{x \in D, |x| \to \infty} p_t^D 1(x) = 0$ for any $t > 0$.*

Proof. Note that for $t > 0$

$$\int_0^t 1_{D \cap B(x,1)^c}(B_s) ds \le \int_0^t 1_{B(x,1)^c}(B_s) ds \le (t - \tau_{B(x,1)})^+$$

($a^+ = a \vee 0$) and that

$$\{\tau_D > t\} \subset \left\{ \int_0^t 1_D(B_s) ds = t \right\}.$$

We then have for any $0 < \varepsilon < t$

$$\begin{aligned}
\mathbb{P}_x(\tau_D > t) &\le \mathbb{P}_x\left(\int_0^t 1_{D \cap B(x,1)}(B_s) ds \ge \varepsilon \right) + \mathbb{P}_x\left(\int_0^t 1_{D \cap B(x,1)^c}(B_s) ds \ge t - \varepsilon \right) \\
&\le \mathbb{P}_x\left(\int_0^t 1_{D \cap B(x,1)}(B_s) ds \ge \varepsilon \right) + \mathbb{P}_x\left((t - \tau_{B(x,1)})^+ \ge t - \varepsilon \right).
\end{aligned}$$

By Lemma 3.1

$$\mathbb{E}_x\left(\int_0^t 1_{D \cap B(x,1)}(B_s) ds \right) \le \int_0^t \frac{C}{s^{d/(2p)}} ds \cdot m(D \cap B(x,1))^{1/p},$$

and $\int_0^t 1/s^{d/(2p)} ds < \infty$ for $p > d/2$. Hence the right-hand side above tends to 0 as $|x| \to \infty$ in D by the assumption on D, and thus

$$\lim_{x \in D, |x| \to \infty} \mathbb{P}_x\left(\int_0^t 1_{D \cap B(x,1)}(B_s) ds \ge \varepsilon \right) = 0.$$

Noting that $\mathbb{P}_x\left((t - \tau_{B(x,1)})^+ \ge t - \varepsilon \right) = \mathbb{P}_0(\tau_{B(1)} \le \varepsilon)$, we have

$$\limsup_{x \in D, |x| \to \infty} p_t^D 1(x) = \limsup_{x \in D, |x| \to \infty} \mathbb{P}_x(\tau_D > t) \le \mathbb{P}_0(\tau_{B(1)} \le \varepsilon) \to 0$$

as $\varepsilon \to 0$. □

From Lemma 3.2, we immediately obtain the next corollary.

Corollary 3.1. *If a domain D belongs to $\mathcal{D}_0$, then $\lim_{x \in D, |x| \to \infty} R_1^D 1(x) = 0$.*

Lemma 3.3. *If a domain D belongs to $\mathcal{D}_0$, then the absorbing BM on D is in Class (T).*

Proof. The irreducibility I and the resolvent strong Feller property II follow from [5, Exercise 4.6.3] and [3, Theorem 2.2] respectively.

Note that for a compact subset K of D

$$R_1^D 1_{K^c} = R_1^D 1_{B(R)^c \cap K^c} + R_1^D 1_{B(R)\cap K^c} \le R_1^D 1_{B(R)^c\cap K^c} + R_1 1_{D\cap B(R)\cap K^c}.$$

We see that by the maximum principle

$$\sup_{x\in D} R_1^D 1_{B(R)^c\cap K^c}(x) = \sup_{x\in D\cap B(R)^c} R_1^D 1_{B(R)^c\cap K^c}(x) \le \sup_{x\in D\cap B(R)^c} R_1^D 1(x)$$

and that by Corollary 3.1 the right-hand side above tends to 0 as $R \to \infty$. Hence, for any $\epsilon > 0$ there exists $R > 0$ such that $\sup_{x\in D} R_1^D 1_{B(R)^c\cap K^c}(x) \le \epsilon/2$ for any compact subset $K \subset D$.

Let $\{K_n\}_{n=1}^\infty$ be an increasing sequence of compact subsets of $D \cap B(R)$ such that $\lim_{n\to\infty} m(D \cap B(R) \cap K_n^c) = 0$. Then by using the maximum principle again

$$\lim_{n\to\infty} \sup_{x\in D} R_1 1_{D\cap B(R)\cap K_n^c}(x) = \lim_{n\to\infty} \sup_{x\in\mathbb{R}^d} R_1 1_{D\cap B(R)\cap K_n^c}(x) = 0.$$

Hence, $\sup_{x\in D} R_1 1_{D\cap B(R)\cap K_n^c}(x) \le \epsilon/2$ for a large n. Therefore, the tightness property III of the absorbing BM on D is proved. □

We now obtain the next corollary as an application of Theorem 2.1.

Corollary 3.2. *If a domain D belongs to $\mathcal{D}_0$, then the semi-group of the Dirichlet Laplacian on D is compact.*

4. Compact Embedding of the Sobolev Spaces

At the first part of this section, the 1-resolvent is associated with the d-dimensional Brownian motion.

We set

$$\mathcal{B}_0 = \left\{ B \in \mathcal{B}(\mathbb{R}^d) \mid \lim_{|x|\to\infty} m(B \cap B(x, 1)) = 0 \right\}.$$

Note that for $B \in \mathcal{B}_0$

$$\lim_{|x|\to\infty} m(B \cap B(x, R)) = 0, \quad \forall R > 0. \tag{4.1}$$

The 1-resolvent kernel of the d-dimensional Brownian motion ($d \ge 3$) has the following bound ([9, Theorem 6.23])[2]:

$$R_1(x, y) \simeq \begin{cases} \dfrac{1}{|x-y|^{d-2}}, & |x-y| \le 1, \\ \dfrac{e^{-\sqrt{2}|x-y|}}{|x-y|^{(d-1)/2}}, & |x-y| \ge 1. \end{cases}$$

Lemma 4.1. *B belongs to $\mathcal{B}_0$ if and only if $m^B(\bullet)(= m(B \cap \bullet))$ is in $\mathcal{K}_\infty(R_1)$.*

[2]For positive functions $f(z)$ and $g(z)$ on some set Z, we write $f \simeq g$ if there exists a positive constant C such that $C^{-1} \le f(z)/g(z) \le C, \ \forall z \in Z$.

Proof. Suppose $B \in \mathcal{B}_0$. For $R > l > 1$

$$B_1 = B(R)^c \cap B(x,l) \cap B, \quad B_2 = B(R)^c \cap B(x,l)^c \cap B.$$

Since $B(R)^c \cap B(x,l) = \emptyset$ for $x \in B(R-l)$, we have

$$\begin{aligned} R_1 1_{B(R)^c \cap B}(x) &\le C_1 \int_{B_1} \frac{1}{|x-y|^{d-2}} dy + C_2 \int_{B_2} \frac{e^{-\sqrt{2}|x-y|}}{|x-y|^{(d-1)/2}} dy \\ &\le C_1 \sup_{x \in B(R-l)^c} \int_{B(x,l)\cap B} \frac{1}{|x-y|^{d-2}} dy + C_2 \int_{B(x,l)^c} \frac{e^{-\sqrt{2}|x-y|}}{|x-y|^{(d-1)/2}} dy. \end{aligned}$$

For any $\varepsilon > 0$, the second term of the right-hand side is less than $\varepsilon/2$ for a large l, and the first term is less than $\varepsilon/2$ for a large R because $B \in \mathcal{B}_0$. Hence m^B belongs to $\mathcal{K}_\infty(R_1)$.

Suppose $m^B \in \mathcal{K}_\infty(R_1)$. Then for $x \in B(R+1)^c$

$$\begin{aligned} \int_{B(R)^c} R_1(x,y) m^B(dy) &\ge \int_{B(R)^c \cap B \cap B(x,1)} R_1(x,y) dy \ge c_1 \int_{B \cap B(x,1)} \frac{1}{|x-y|^{d-2}} dy \\ &\ge c_1 m(B \cap B(x,1)), \end{aligned}$$

and thus

$$\limsup_{R\to\infty} \sup_{|x| \ge R+1} m(B \cap B(x,1)) \le \lim_{R\to\infty} \sup_{x \in \mathbb{R}^d} \int_{B(R)^c} R_1(x,y) m^B(dy) = 0.$$

□

We obtain the next corollary from Corollary 2.1.

Corollary 4.1. *If $B \in \mathcal{B}_0$, then $H^1(\mathbb{R}^d)$ is compactly embedded in $L^2(B)$.*

Corollary 4.1 is known (cf. [4, Chapter X, Lemma 6.11, Lemma 6.12]). Moreover, it is shown in [4, Lemma 6.11] that the condition for an open set D being in $\mathcal{B}_0$ is a necessary and sufficient one for $H^1(\mathbb{R}^d)$ being compactly embedded in $L^2(D)$. Hence we can summarize as follows:

Theorem 4.1. *Let D be a domain of $\mathbb{R}^d$. The following statements are equivalent.*

(i) $D \in \mathcal{B}_0$;
(ii) $m^D \in \mathcal{K}_\infty(R_1)$;
(iii) $H^1(\mathbb{R}^d)$ *is compactly embedded in* $L^2(D)$.

4.1. Existence of Ground States. In the sequel, let us consider the symmetric α-stable process on $\mathbb{R}^d$, the Lévy process with generator $-(-\Delta)^{\alpha/2}$, $0 < \alpha \le 2$, and denote it by $X^\alpha = (\mathbb{P}_x, X_t)$. We suppose, in addition, the transience of X^α, $d > \alpha$. The Dirichlet form $(\mathcal{E}, \mathcal{F})$ of X^α on $L^2(\mathbb{R}^d)$ is expressed by

$$\mathcal{E}^\alpha(u,v) = \frac{1}{2} \iint_{\mathbb{R}^d \times \mathbb{R}^d} (u(y)-u(x))(v(y)-v(x)) \frac{A_{d,\alpha}}{|x-y|^{d+\alpha}} dxdy, \quad \mathcal{F} = H^{\alpha/2}(\mathbb{R}^d),$$

where $H^{\alpha/2}(\mathbb{R}^d)$ is the Sobolev space of order $\alpha/2$ and

$$A_{d,\alpha} = \frac{\alpha \cdot 2^{\alpha-1}\Gamma(\frac{\alpha+d}{2})}{\pi^{\frac{d}{2}}\Gamma(1-\frac{\alpha}{2})}, \qquad \Gamma(s) := \int_0^\infty x^{s-1} e^{-x} dx.$$

The transition density of X^α, $p(t,x,y)$, satisfies

$$p(t,x,y) \simeq t^{-\frac{d}{\alpha}} \wedge \frac{t}{|x-y|^{d+\alpha}}, \tag{4.2}$$

and the 1-resolvent density $R_1(x,y)$

$$R_1(x,y) \simeq \int_0^{|x-y|^\alpha} e^{-t}\frac{t}{|x-y|^{d+\alpha}}dt + \int_{|x-y|^\alpha}^\infty e^{-t}t^{-\frac{d}{\alpha}}dt.$$

The first trem of the right-hand side above equals

$$\frac{1-(1+|x-y|^\alpha)e^{-|x-y|^\alpha}}{|x-y|^{d+\alpha}} \simeq \begin{cases} \dfrac{1}{|x-y|^{d-\alpha}}, & |x-y| \le 1, \\ \dfrac{1}{|x-y|^{d+\alpha}}, & |x-y| \ge 1, \end{cases}$$

and the second term is less than

$$e^{-|x-y|^\alpha}\int_{|x-y|^\alpha}^\infty t^{-\frac{d}{\alpha}}dt = \frac{\alpha}{d-\alpha}\frac{e^{-|x-y|^\alpha}}{|x-y|^{d-\alpha}}.$$

We then see that

$$R_1(x,y) \simeq \begin{cases} \dfrac{1}{|x-y|^{d-\alpha}}, & |x-y| \le 1, \\ \dfrac{1}{|x-y|^{d+\alpha}}, & |x-y| \ge 1. \end{cases} \tag{4.3}$$

For $V \in \mathcal{B}_+(\mathbb{R}^d)$ let

$$M(r) = \operatorname*{ess\,sup}_{x\in B(r)^c} V(x)$$

and set

$$\mathcal{V} = \left\{ V \in \mathcal{B}_+(\mathbb{R}^d) \mid \lim_{|x|\to\infty} \|V1_{B(x,1)}\|_1 = 0,\ \exists r_0 > 0 \text{ s.t. } M(r_0) < \infty \right\}.$$

Corollary 4.2. *If V is in $\mathcal{V} \cap \mathcal{K}$, then V belongs to $\mathcal{K}_\infty(R_1)$. In particular, if $B \in \mathcal{B}(\mathbb{R}^d)$ is in $\mathcal{B}_0$, then $H^\beta(\mathbb{R}^d)$, $0 < \beta \le 1$ is compactly embedded in $L^2(B)$.*

Let

$$V_\gamma(x) = \frac{1}{|x|^\gamma} \wedge 1, \quad \gamma > 0.$$

Lemma 4.2. *Let R be the Green function of the transient symmetric α-stable process, $R(x,y) \simeq 1/|x-y|^{d-\alpha}$. Then V_γ belongs to $\mathcal{K}_\infty(R)$ if and only if $\gamma > \alpha$.*

Proof. If $\gamma > \alpha$, then $V_\gamma \in \mathcal{K}_\infty(R)$. Indeed, take p so that $d/\alpha > p > d/\gamma$ and let $q = p/(p-1)$. For $R > 1 > \varepsilon > 0$

$$\begin{aligned} \int_{\{|y|\ge R\}\cap\{|y-x|\ge\varepsilon\}} \frac{1}{|x-y|^{d-\alpha}}\frac{1}{|y|^\gamma}dy &\le \left(\int_{\{|y-x|\ge\varepsilon\}} \frac{1}{|x-y|^{(d-\alpha)q}}\right)^{1/q}\left(\int_{\{|y|\ge R\}} \frac{1}{|y|^{\gamma p}}dy\right)^{1/p} \\ &= \omega_1\left(\int_\varepsilon^\infty \frac{1}{r^{(d-\alpha)q-d+1}}dr\right)^{1/q}\left(\int_R^\infty \frac{1}{r^{\gamma p-d+1}}dr\right)^{1/p}. \end{aligned}$$

Since $(d-\alpha)q-d+1=(d-\alpha p)/(p-1)+1>1$ and $\gamma p-d+1>1$, the right hand side tends to 0 as $R\to\infty$. Hence

$$\limsup_{R\to\infty}\sup_{x\in\mathbb{R}^d}\int_{\{|y|\geq R\}}\frac{1}{|x-y|^{d-\alpha}}V_\gamma(y)dy\leq\sup_{x\in\mathbb{R}^d}\int_{\{|y|\geq R\}\cap\{|y-x|\leq\varepsilon\}}\frac{1}{|x-y|^{d-\alpha}}V_\gamma(y)dy.$$

Since

$$\lim_{\varepsilon\to 0}\sup_{x\in\mathbb{R}^d}\int_{\{|y|\geq R\}\cap\{|y-x|\leq\varepsilon\}}\frac{1}{|x-y|^{d-\alpha}}V_\gamma(y)dy\leq\lim_{\varepsilon\to 0}\sup_{x\in\mathbb{R}^d}\int_{\{|y-x|\leq\varepsilon\}}\frac{1}{|x-y|^{d-\alpha}}dy=0,$$

we have

$$\lim_{R\to\infty}\sup_{x\in\mathbb{R}^d}\int_{\{|y|\geq R\}}\frac{1}{|x-y|^{d-\alpha}}V_\gamma(y)dy=0.$$

For $\gamma\leq\alpha$

$$\begin{aligned}\sup_{x\in\mathbb{R}^d}\int_{\{|y|\geq R\}}\frac{1}{|x-y|^{d-\alpha}}V_\gamma(y)dy&\geq\int_{\{|y|\geq R\}}\frac{1}{|y|^{d-\alpha}}\frac{1}{|y|^\gamma}dy\\&=\omega_1\int_R^\infty\frac{1}{r^{\gamma-\alpha+1}}dr=\infty,\end{aligned}$$

and thus $V_\gamma\notin\mathcal{K}_\infty(R)$. □

For any $\gamma>0$, V_γ belongs to $\mathcal{K}\cap\mathcal{V}$ and so to $\mathcal{K}_\infty(R_1)$ by Corollary 4.2. The lemma above tells us that $\mathcal{K}_\infty(R)$ is strictly included in $\mathcal{K}_\infty(R_1)$.

We see that for $\alpha<\gamma\leq d$, V_γ is in $\mathcal{K}_\infty(R)$ with $\int_{\mathbb{R}^d}V_\gamma(x)dx=\infty$. Combining Theorem 2.3 with Lemma 4.2, we see that if $\gamma>\alpha$, then the extended Dirichlet space $H_e^{\alpha/2}(\mathbb{R}^d)$ is compactly embedded in $L^2(\mathbb{R}^d;V_\gamma dx)$. However, we see that the embedding is not compact if $\gamma=\alpha$. Indeed, we see from Hardy's inequality,

$$\int_{\mathbb{R}^d}u^2(x)\frac{1}{|x|^\alpha}dx\leq C\mathcal{E}^\alpha(u,u)$$

that $H_e^{\alpha/2}(\mathbb{R}^d)$ is continuously embedded in $L^2(\mathbb{R}^d;V_\alpha dx)$. In other words, the 0-order resolvent operator $\check{R}$ of the time-changed process by $\int_0^t V_\alpha(X_s)ds$,

$$\check{R}^\alpha f(x)=R(V_\alpha f)(x)$$

is a bounded operator on $L^2(\mathbb{R}^d;V_\alpha dx)$ and so is

$$T^\alpha f(x):=\int_{\mathbb{R}^d}K^\alpha(x,y)f(y)dy,\quad K^\alpha(x,y)=\frac{\sqrt{V_\alpha(x)V_\alpha(y)}}{|x-y|^{d-\alpha}}$$

on $L^2(\mathbb{R}^d)$ because of the unitary equivalence between $\check{R}^\alpha$ and T^α. Moreover, the compact embedding of $H_e^{\alpha/2}(\mathbb{R}^d)$ into $L^2(\mathbb{R}^d;V_\alpha dx)$ is equivalent to the compactness of the operator T^α on $L^2(\mathbb{R}^d)$. The kernel K^α is called the *Birman-Schwinger Kernel* (cf. [12, Section 7.9]). Note that the time changed operator $\check{R}$ can be defined for a smooth measure μ by $R^\alpha(f\mu)$; however, T^α cannot be defined because the root of measure μ has no meaning.

Let $\varphi_0=1_{B(2)\setminus B(1)}$ and define

$$\varphi_n(x)=2^{-\frac{d(d-\alpha)}{2}n}\varphi_0(2^{-(d-\alpha)n}x).$$

Then we can check that $\|\varphi_n\|_2 = \|\varphi_0\|_2$, φ_n converges L^2-weakly to 0, and

$$(\varphi_n, T^\alpha \varphi_n) = \iint_{1\le|x|\le 2, 1\le|y|\le 2} \frac{1}{|x|^{\alpha/2}|x-y|^{d-\alpha}|y|^{\alpha/2}} dxdy.$$

If T^α is compact, then $T^\alpha\varphi_n$ converges L^2-strongly to 0 and $(\varphi_n, T^\alpha\varphi_n)$ converges to 0 as $n \to \infty$, which is contradictory. Hence, we have the next proposition.

Proposition 4.1. *Suppose $d > \alpha$. The extended Dirichlet space $H_e^{\alpha/2}(\mathbb{R}^d)$ is compactly embedded in $L^2(\mathbb{R}^d; V_\gamma dx)$ if and only if $\gamma > \alpha$.*

Using Corollary 2.1, we show existence of ground states of Schrödinger operators. There exists a decreasing function g on $[0, \infty)$ and $R_1(x, y)$ is written as

$$R_1(x, y) = g(|x - y|).$$

and for $V \in \mathcal{K} \cap L^1(\mathbb{R}^d)$

$$\begin{aligned}\int_{\mathbb{R}^d} R_1(x,y)V(y)dy &= \int_{|x-y|\le\varepsilon} g(|x-y|)V(y)dy + \int_{|x-y|>\varepsilon} g(|x-y|)V(y)dy \\ &\le k(\varepsilon) + g(\varepsilon)\|V\|_1,\end{aligned} \tag{4.4}$$

where

$$k(\varepsilon) = \sup_{x\in\mathbb{R}^d} \int_{|x-y|\le\varepsilon} g(|x-y|)V(y)dy.$$

It is known in [1] that

$$V \in \mathcal{K} \iff \lim_{\varepsilon\downarrow 0} k(\varepsilon) = 0. \tag{4.5}$$

Lemma 4.1 can be extended as follows:

Proposition 4.2. *If V is in $\mathcal{V} \cap \mathcal{K}$, then V belongs to $\mathcal{K}_\infty(R_1)$.*

Proof. For $R > l > r_0$,

$$\begin{aligned}\int_{B(R)^c} R_1(x,y)V(y)dy &= \int_{B(R)^c\cap B(x,l)^c} g(|x-y|)V(y)dy + \int_{B(R)^c\cap B(x,l)} g(|x-y|)V(y)dy \\ &\le M(r_0)\omega_1 \int_l^\infty g(r)r^{d-1}dr + \int_{B(R)^c} g(|x-y|)(V1_{B(x,l)})(y)dy,\end{aligned}$$

where ω_1 is the surface area of the unit sphere. By (4.4) the second term of the right-hand side is less than

$$\sup_{x\in B(R-l)^c} \int_{\mathbb{R}^d} g(|x-y|)(V1_{B(x,l)})(y)dy \le \sup_{x\in B(R-l)^c} (k(\varepsilon) + g(\varepsilon)\|V1_{B(x,l)}\|_1).$$

By the assumption $V \in \mathcal{V}$,

$$\limsup_{R\to\infty} \sup_{x\in\mathbb{R}^d} \int_{B(R)^c} R_1(x,y)V(y)dy \le M(r_0)\omega_1 \int_l^\infty g(r)r^{d-1}dr + k(\varepsilon)$$

and by (4.5) the second term of the right-hand side tends to 0 as $\varepsilon \downarrow 0$. Letting $l \uparrow \infty$ leads us to $V \in \mathcal{K}_\infty(R_1)$. □

Note that the equivalence (4.5) is valid for X^α(cf. [7]). Then the estimate (4.3) of R_1 leads us to Proposition 4.2 for X^α by the same argument.

Proposition 4.3. *If $V \in \mathcal{K}$ satisfies $V1_{\{V \geq \varepsilon\}} \in L^1(\mathbb{R}^d)$ for any $\varepsilon > 0$, then $V \in \mathcal{K}_\infty(R_1)$.*

Proof. Since the 1-resolvent kernel $R_1(x, y)$ can be written as $g(|x-y|)$,

$$\int_{B(R)^c} R_1(x,y)V(y)dy = \int_{B(R)^c \cap \{V \geq \varepsilon\}} g(|x-y|)V(y)dy + \int_{B(R)^c \cap \{V < \varepsilon\}} g(|x-y|)V(y)dy$$
$$\leq \int_{B(R)^c \cap \{V \geq \varepsilon\}} g(|x-y|)V(y)dy + \varepsilon\omega_1 \int_0^\infty g(r)r^{d-1}dr.$$

Noting $\mathcal{K} \cap L^1(\mathbb{R}^d) \subset \mathcal{K}_\infty(R)$ by [16, Proposition 1], we have

$$\limsup_{R\to\infty} \sup_{x\in\mathbb{R}^d} \int_{B(R)^c} R_1(x,y)V(y)dy \leq \varepsilon\omega_1 \int_0^\infty g(r)r^{d-1}dr \longrightarrow 0, \quad \varepsilon \downarrow 0.$$

□

For $V = V^+ - V^- \in \mathcal{K}_{loc} - \mathcal{K}$ we define

$$\mathcal{E}^V(u,u) = \frac{1}{2}\mathbb{D}(u,u) + \int_{\mathbb{R}^d} u^2 V dx, \quad u \in H^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d; V^+dx),$$

where $\mathbb{D}$ denotes the Dirichlet integral.

Corollary 4.3. *Let $V = V^+ - V^- \in \mathcal{K}_{loc} - \mathcal{K} \cap \mathcal{V}$. If*

$$\lambda_0 := \inf\left\{\mathcal{E}^V(u,u) \mid u \in H^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d : V^+dx), \int_{\mathbb{R}^d} u^2 dx = 1\right\} < 0, \tag{4.6}$$

then a minimizer for λ_0 exists.

Proof. Let γ_0 be the positive constant such that

$$\inf\left\{\mathcal{E}^{V^+}(u,u) + \gamma_0(u,u)_m \mid \int_{\mathbb{R}^d} u^2 V^- dx = 1,\ u \in H^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d; V^+dx)\right\} = 1. \tag{4.7}$$

V^- belongs to $\mathcal{K}_\infty(R_1) \subset \mathcal{K}_\infty(R_1^{V^+})$ by Proposition 4.2 and a minimizer, φ_0, in (4.7) exists by Corollary 2.1. Put $\phi_0 = \varphi_0/\|\varphi_0\|_2$. Then $\|\phi_0\|_2 = 1$, $\mathcal{E}^V(\phi_0,\phi_0) + \gamma_0(\phi_0,\phi_0)_m = 0$ and thus

$$\inf\left\{\mathcal{E}^V(u,u) + \gamma_0(u,u)_m \mid \int_{\mathbb{R}^d} u^2 dx = 1,\ u \in H^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d; V^+dx)\right\} \leq 0. \tag{4.8}$$

We see from the same argument as in [15, Lemma 2.2] that

$$\inf\left\{\mathcal{E}^{V^+}(u,u) + \gamma_0(u,u)_m \mid \int_{\mathbb{R}^d} u^2 V^- dx = 1,\ u \in H^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d; V^+dx)\right\} \geq 1$$

if and only if

$$\inf\left\{\mathcal{E}^V(u,u) + \gamma_0(u,u)_m \mid \int_{\mathbb{R}^d} u^2 dx = 1,\ u \in H^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d; V^+dx)\right\} \geq 0.$$

Hence by combing (4.7) with (4.8) we conclude that

$$\gamma_0 + \inf\left\{\mathcal{E}^V(u,u) \mid \int_{\mathbb{R}^d} u^2 dx = 1,\ u \in H^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d; V^+dx)\right\} = 0,$$

λ_0 equals $-\gamma_0$ and $\varphi_0/\|\varphi_0\|_2$ is a minimizer for λ_0. □

Suppose that $V \in L^{d/2}(\mathbb{R}^d) + L^\infty(\mathbb{R}^d)$ on $\mathbb{R}^d$ *vanishes at infinity*, that is, it satisfies

$$m(\{x \mid |V(x)| > \varepsilon\}) < \infty \quad \text{for all } \varepsilon > 0. \tag{4.9}$$

Then it is known in [9, Theorem 11.5] that if, in addition, V satisfies (4.6), then a minimizer exists. Note that $V \in \mathcal{V}$ does not satisfy (4.9) in general. Indeed, for $B \in \mathcal{B}_0$ with $m(B) = \infty$, $V := 1_B$ does not satisfy (4.9).

5. Schrödinger Semi-groups

Recall that E is a locally compact separable metric space and m is a positive Radon measure on E with full support. Let X be an m-symmetric Borel right process having left limits on $(0, \zeta)$, where ζ is the life time (see section 2). In this section, we assume that X has the properties I and II. We define the Schrödinger semi-group $\{p_t^\mu\}_{t\geq 0}$ by

$$p_t^\mu f(x) = \mathbb{E}_x\left(e^{-A_t^\mu} f(X_t)\right), \quad f \in \mathcal{B}_b(E),$$

and consider the compactness of the operator p_t^μ on $L^2(E; m)$.

Lemma 5.1. $\lim_{x\to\infty} R_1^\mu 1(x) = 0$ *if and only if* $\lim_{x\to\infty} p_t^\mu 1(x) = 0$ *for any* $t > 0$.

Proof. The "if" part is clear. Noting

$$R_1^\mu 1(x) = \int_0^\infty e^{-s} p_s^\mu 1(x) ds \geq \int_0^t e^{-s} p_s^\mu 1(x) ds \geq t e^{-t} p_t^\mu 1(x),$$

we have this lemma. □

A measure μ is said to be in $\mathcal{K}_{loc}$ if $1_G \mu$ is of Kato class for any relatively compact open set $G \subset E$.

Theorem 5.1. *Let $\mu \in \mathcal{K}_{loc}$. Assume that for any $M > 0$ there exists a Borel set D_M such that*

(i) *$\mu \geq M \cdot m$ on D_M^c,*
(ii) *for any $t > 0$ and any $\epsilon > 0$*

$$\lim_{|x|\to\infty} \mathbb{P}_x\left(\int_0^t 1_{D_M}(X_s) ds > \epsilon,\ t < \zeta\right) = 0.$$

Then p_t^μ is compact.

Proof. Owing to Remark 2.1 (i) and Lemma 5.1, it is sufficient to show that $\lim_{x\to\infty} p_t^\mu 1(x) = 0$ for any $t > 0$.

Since

$$\left\{\omega \in \Omega \mid \int_0^t 1_{D_M^c}(X_s) ds \geq t - \epsilon,\ t < \zeta\right\} = \left\{\omega \in \Omega \mid \int_0^t 1_{D_M}(X_s) ds \leq \epsilon,\ t < \zeta\right\},$$

we have

$$
\begin{aligned}
p_t^\mu 1(x) &= \mathbb{E}_x\left(e^{-A_t^\mu};\int_0^t 1_{D_M}(X_s)ds > \epsilon,\ t<\zeta\right) \\
&\quad + \mathbb{E}_x\left(e^{-A_t^\mu};\int_0^t 1_{D_M^c}(X_s)ds \geq t-\epsilon,\ t<\zeta\right).
\end{aligned} \tag{5.1}
$$

It follows from the assumption (i) that if $\int_0^t 1_{D_M^c}(X_s)ds \geq t-\epsilon$, then $\int_0^t 1_{D_M^c}(X_s)dA_s^\mu \geq M(t-\epsilon)$. Hence the second term of (5.1) is less than $\exp(-M(t-\epsilon))$ and thus

$$
\limsup_{|x|\to\infty} p_t^\mu 1(x) \leq e^{-M(t-\epsilon)}
$$

by the assumption (ii). We have the desired claim by letting M to ∞. □

In the sequel, let us consider the symmetric α-stable process on $\mathbb{R}^d$, the Lévy process with generator $-(-\Delta)^{\alpha/2}$, $0<\alpha\leq 2$, and denote it by $X^\alpha = (\mathbb{P}_x, X_t)$. Let V be a positive function on $\mathbb{R}^d$ in the local Kato class. Set

$$
V_M = \{x\in\mathbb{R}^d \mid V(x)\leq M\}.
$$

Lemma 5.2. *If $V_M \in \mathcal{B}_0$, then*

$$
\lim_{|x|\to\infty} \mathbb{P}_x\left(\int_0^t 1_{V_M}(X_s)ds > \epsilon\right) = 0.
$$

Proof. We have

$$
\begin{aligned}
\mathbb{P}_x\left(\int_0^t 1_{V_M}(X_s)ds > \epsilon\right) &= \mathbb{P}_x\left(\int_0^t 1_{V_M\cap B(x,R)}(X_s)ds + \int_0^t 1_{V_M\cap B(x,R)^c}(X_s)ds > \epsilon\right) \\
&\leq \mathbb{P}_x\left(\int_0^t 1_{V_M\cap B(x,R)}(X_s)ds > \frac{\epsilon}{2}\right) + \mathbb{P}_x\left(\int_0^t 1_{V_M\cap B(x,R)^c}(X_s)ds > \frac{\epsilon}{2}\right).
\end{aligned} \tag{5.2}
$$

Note that by the same argument as in Lemma 3.1, the semi-group p_t of X^α satisfies $\|p_t\|_{p,\infty} \leq C/t^{d/(\alpha p)}$. We then see that for $p > d/\alpha$ the first term of the right-hand side is dominated by

$$
\frac{2}{\epsilon}\mathbb{E}_x\left(\int_0^t 1_{V_M\cap B(x,R)}(X_s)ds\right) \leq C(\varepsilon,t)\cdot m(V_M\cap B(x,R))^{1/p}
$$

and tends to 0 as $|x|\to\infty$ on account of (4.1), where m means the Lebesgue measure on $\mathbb{R}^d$.

Since $\int_0^t 1_{V_M\cap B(x,R)^c}(X_s)ds \leq (t-\tau_{B(x,R)})^+$, the second term of the right-hand side of (5.2) is dominated by

$$
\mathbb{P}_x(t-\tau_{B(x,R)} > \epsilon/2) = \mathbb{P}_0(\tau_{B(R)} < t-\epsilon/2) \longrightarrow 0
$$

as $R\to\infty$. Here $\tau_{B(x,R)}$ is the first exist time from $B(x,R)$. Therefore, we have this lemma. □

Lemma 5.2 is valid for any $B\in\mathcal{B}_0$. Combining Theorem 5.1 with Lemma 5.2, we have the next theorem.

Theorem 5.2. *Let $V\in\mathcal{K}_{loc}$. If $V_M\in\mathcal{B}_0$ for any $M>0$, then the semi-group of $(-\Delta)^{\alpha/2}+V$ is compact.*

For the symmetric α-stable process, the compactness of p_t^V is equivalent to $\lim_{|x|\to\infty} p_t^V 1(x) = 0$ ([6, Lemma 9]). On account of Remark 2.1 (ii) and Lemma 5.1 we have the next corollary.

Corollary 5.1. *For $V \in \mathcal{K}_{loc}$, let X^V be the subprocess of the symmetric α-stable process by the multiplicative functional* $\exp(-\int_0^t V(X_s)ds)$. *Then the following statements are equivalent.*

(i) *X^V is in Class (T);*
(ii) $\lim_{|x|\to\infty} p_t^V 1(x) = 0$*;*
(iii) *p_t^V is compact on* $L^2(\mathbb{R}^d)$.

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